Stochastic Selfish Routing

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We present a model for routing games in which edge delay functions are uncertain and users are risk-averse. We investigate how the uncertainty and risk-aversion transform the classical theory on routing games, including equilibria existence, characterization and price of anarchy.

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1. INTRODUCTION

Routing games were one of the central examples in the development of algorithmic game theory. In these games, multiple users need to route between different source-destination pairs and edges are congestible, namely, each edge delay $l_e(x)$ is a non-decreasing function of the flow or number of users x on the edge. Many of the fundamental game theoretic questions are now well understood for these games, for example, does equilibrium exist, is it unique, can it be computed efficiently, does it have a compact representation; the same questions can be asked of the socially optimal solution that minimizes the total user delay. Furthermore, routing games were a primary motivation and application for the study of the price of anarchy, which quantifies the inefficiency of equilibria.

So far, most research has focused on the classical models in which the edge delays are deterministic. In contrast, real world applications contain a lot of uncertainty, which may stem from exogenous factors such as weather, time of day, weekday versus weekend, etc. or endogenous factors such as the network traffic. Furthermore, many users are risk-averse in the presence of uncertainty, so that they do not simply want to minimize expected delays and instead may need to add a buffer to ensure a guaranteed arrival time to a destination. This fundamentally changes the mathematical structure of the routing problems and, consequently, the behavior and properties in routing games as well.

Our work [Nikolova and Stier-Moses 2011; 2012] aims to initiate a theoretical study of how uncertainty and risk aversion transform the classical theory of routing games. Integrating different models of uncertainty and different measures of risk can easily fill a long-term research agenda. We therefore focus on the special model

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defined in the next section and hope to motivate other researchers to join in the effort by considering generalizations or alternative models.

2. MODEL

Consider a directed graph $G = (V, E)$ with an aggregate demand of d_k units of flow between source-destination pairs (s_k, t_k) for $k \in K$. We let $\mathcal{P}_k$ be the set of all paths between s_k and t_k, and $\mathcal{P} := \cup_{k \in K} \mathcal{P}_k$ be the set of all paths. We encode players decisions as a flow vector $\mathbf{f} = (f_\pi)_{\pi \in \mathcal{P}} \in \mathbb{R}_+^{|\mathcal{P}|}$ over all paths. Such a flow is feasible when demands are satisfied, as given by constraints $\sum_{\pi \in \mathcal{P}_k} f_\pi = d_k$ for all $k \in K$. The congestible network is modeled with stochastic delay functions $\ell_e(x_e) + \xi_e(x_e)$ for each edge $e \in E$. Here, $\ell_e(x_e)$ measures the expected delay when the edge has flow x_e, and the random variable $\xi_e(x_e)$ represents the stochastic delay error. The function $\ell_e(\cdot)$ is assumed continuous and non-decreasing, $\mathbb{E}(\xi_e(x_e)) = 0$, and $\mathrm{Stdev}(\xi_e(x_e)) = \sigma_e(x_e)$, for a continuous function $\sigma_e(\cdot)$. Although the distribution of delay may depend on the flow x_e, we separately consider the simplified case in which $\sigma_e(x_e) = \sigma_e$ is a constant given exogenously, independent from x_e. We also assume that delays are uncorrelated with each other (see [Nikolova 2009], p. 96 for a discussion on how to incorporate local correlations).

Risk-averse players choose paths according to a mean-stdev objective, which we refer to as the cost along route π:

$$Q_\pi(\mathbf{f}) := \sum_{e \in \pi} \ell_e\Big(\sum_{p: e \in p} f_p\Big) + \gamma \sqrt{\sum_{e \in \pi} \sigma_e\Big(\sum_{p: e \in p} f_p\Big)^2}, \tag{1}$$

where $\gamma \geq 0$ quantifies the risk aversion of players, assumed homogeneous.

Adding a constant number of standard deviations to the expectation of delay is a natural approach for adding a buffer to increase the reliability of a route. A compelling interpretation of this objective in the case of normally-distributed uncertainty is that the mean-stdev of a path equals a percentile of delay along it. This model is also related to typical quantifications of risk, most notably the value-at-risk objective commonly used in finance, whereby one seeks to minimize commute time subject to arriving on time to a destination with at least, say, 95% chance. The mean-stdev risk measure has also been used by transportation practitioners, who base the definition of the travel time reliability index on it [Schrank et al. 2010]. At the same time, the measure has been criticized for sometimes preferring a strictly dominated solution.[1] In essence, the objective has a preference for more certain routes, which in fact may be an advantage in applications such as telecommunication networks (voice or video streaming), transportation (intercity bus routes), task planning, robot motion planning, etc. Further discussion of the objective can be found in our work [Nikolova and Stier-Moses 2011; 2012].

[1]For instance, between choosing a path that always takes 1 hr vs a path that takes 1 hr or 50 min with probability $\frac{1}{2}$ each, the objective may prefer the first path even though it is stochastically dominated, since the second path is penalized for its variability through the standard deviation term in the objective.

	Exogenous Standard Deviations	Endogenous Standard Deviations
Nonatomic Users	Equilibrium exists (exponentially-large convex program)	Equilibrium exists (variational inequality)
Atomic Users	Equilibrium exists (potential game)	No pure strategy equilibrium

Table I. Existence of equilibria in mean-risk stochastic selfish routing games.

3. RESULTS

We generalize the traditional model of Wardrop competition [Wardrop 1952] by incorporating stochastic travel times. Technically, this model is much harder to analyze than the traditional one because it is *non-additive*, namely the cost of a path is not equal to the sum of costs of edges along the path [Gabriel and Bernstein 1997]. This in turn means that an equilibrium in the stochastic setting does not decompose to equilibria in subnetworks of the given network, leading to computational and structural complications. Depending on the specific details of the application one has in mind, users may be small or large [Harker 1988]. We consider both infinitesimal users, referred to as the *non-atomic* case, as well as users that control a strictly positive demand, referred to as the *atomic* case.

To analyze the problem and to establish the existence of equilibrium, we draw from a diverse spectrum of tools from potential games and convex analysis to the theory of variational inequalities and nonconvex (stochastic) shortest paths. We consider four settings of nonatomic vs. atomic users and exogenous vs. endogenous variability of travel times. Our conclusions and methods are different in each of these settings. In the nonatomic case with standard deviation of travel times given exogenously, we prove that equilibria always exist using a convex problem with exponentially-many variables similar to that of [Ordóñez and Stier-Moses 2010]. The atomic case with exogenous standard deviations is shown to be a potential game and therefore a pure-strategy Nash equilibrium always exists. To characterize the equilibria of the nonatomic version of the problem when the standard deviations of travel times are endogenous, we use a variational inequality formulation [Hartman and Stampacchia 1966; Smith 1979; Dafermos 1980] that draws ideas from the nonlinear complementary problem formulation of [Aashtiani and Magnanti 1981]. In this case, an equilibrium always exists; in fact, not only for our specific mean-stdev objective but also for any general continuous objective. In contrast, the atomic case with endogenous standard deviation does not always admit a pure-strategy Nash equilibrium. We summarize these results in Table I.

Next, we investigate if there is a succinct representation (in terms of a small set of paths) of user and system-optimal flows in the case of non-atomic users with stochastic travel times. Our results here are independent of whether the standard deviations are exogenous or endogenous. We prove that if one is given a solution (either a Wardrop equilibrium or a system optimum) as an edge-flow, not every path decomposition is a solution, in contrast to the deterministic case where every decomposition works. Nevertheless, there is always a succinct solution that uses at most $|E| + |K|$ paths, where E is the set of edges in the network and K is the set of origin-destination pairs. Although the complexity of computing a solution is unknown (actually, even the complexity of computing a single stochastic shortest path is unknown), this result says that there is some hope because at least solutions can be efficiently encoded.

Finally, we analyze the price of anarchy of mean-risk Wardrop equilibria under stochastic travel times with respect to the socially-optimal solution, for the case of nonatomic users. The social optimum is defined as the flow minimizing the total cost incurred by users, as given by their mean-stdev objective. Surprisingly, under exogenous standard deviations, uncertainty and risk aversion do not exacerbate the inefficiency of equilibria. The price of anarchy remains equal to that of deterministic nonatomic games. Namely, it is 4/3 for the case of linear expected travel times [Roughgarden and Tardos 2002] and $(1 - \beta(\mathcal{L}))^{-1}$ for an appropriately defined constant $\beta(\mathcal{L})$ for expected travel time functions in a class $\mathcal{L}$ [Roughgarden 2003; Correa et al. 2004; 2008].

The case of endogenous standard deviations presents a significant additional difficulty that makes the square root terms in different paths interrelated functions of the path flow that cannot be analyzed separately; a general price of anarchy bound for this case remains elusive. Nevertheless, we show that, despite the square root term, the path costs are convex whenever the individual travel times and standard deviations on edges are convex. Consequently, we present sufficient conditions for convexity of the social cost, which are similar to the sufficient conditions for uniqueness of equilibrium in its variational inequality characterization. Unfortunately, these conditions are fragile and in general the social cost will not be convex and may admit a non-connected set of multiple global minima but we can still identify settings where the price of anarchy is 1.

4. OPEN PROBLEMS

From a high-level philosophical perspective, it is intriguing to understand how users make decisions and what are the right risk-aversion models in uncertain settings. For the correct modeling of routing and other games studied in Algorithmic Game Theory and Mechanism Design, it would be beneficial to draw from fields and areas that have a tradition in decision-making under uncertainty such as Expected Utility Theory and alternative Non-Expected Utility Theories (considered at the intersection of psychology and economics), Portfolio Optimization, Operations Research and Finance.

Some concrete questions that arise from our work include:

—What is the complexity of computing an equilibrium when it exists (exogenous standard deviations with atomic or nonatomic players; endogenous standard deviations with nonatomic players)?

—What is the complexity of computing the socially-optimal solution? What is the complexity of computing the socially-optimal flow decomposition if one knows the edge-flow that represents a socially-optimal solution?

—Can there be multiple equilibria in the nonatomic game with endogenous standard deviations?

—What is the price of anarchy for stochastic Wardrop equilibria in the setting of nonatomic games with endogenous standard deviations, for general graphs and general classes of cost functions?

—Can some of the results in this paper be extended to the case of users with heterogenous attitudes toward risk [Ordóñez and Stier-Moses 2010]?

Of course, one could pursue other natural models and player objectives and build upon or complement the theory we have developed here. In particular, our model might be enriched by also considering stochastic demands to make the demand side more realistic.

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