Finding Any Nontrivial Coarse Correlated Equilibrium Is Hard

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One of the most appealing aspects of correlated equilibria and coarse correlated equilibria is that natural dynamics quickly arrive at approximations of such equilibria, even in games with many players. In addition, there exist polynomial-time algorithms that compute exact correlated and coarse correlated equilibria. However, in general these dynamics and algorithms do not provide a guarantee on the *quality* (say, in terms of social welfare) of the resulting equilibrium. In light of these results, a natural question is how *good* are the correlated and coarse correlated equilibria—in terms natural objectives such as social welfare or Pareto optimality—that can arise from any efficient algorithm or dynamics.

We address this question, and establish strong negative results. In particular, we show that in multiplayer games that have a succinct representation, it is NP-hard to compute any coarse correlated equilibrium (or approximate coarse correlated equilibrium) with welfare strictly better than the *worst* possible. The focus on succinct games ensures that the underlying complexity question is interesting; many multiplayer games of interest are in fact succinct. We show that analogous hardness results hold for correlated equilibria, and persist under the egalitarian objective or Pareto optimality.

To complement the hardness results, we develop an algorithmic framework that identifies settings in which we can efficiently compute an approximate correlated equilibrium with near-optimal welfare. We use this framework to develop an efficient algorithm for computing an approximate correlated equilibrium with near-optimal welfare in aggregative games.

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1. INTRODUCTION

Questions related to the complexity of equilibria lie at the core of algorithmic game theory. While computation of Nash equilibria has in recent years been shown to be computationally hard even in games with two players [Chen et al. 2009], algorithmic results for correlated equilibria (CE) and coarse correlated equilibria (CCE) have been more positive. Even in games with many players, there exist a number of natural dynamics that quickly converge to these solution concepts; see, e.g., [Young 2004]. In particular, these dynamics induce efficient computation of approximate CE and CCE in multiplayer games. In fact, exact CE and CCE are efficiently computable in many classes of multiplayer games [Papadimitriou and Roughgarden 2008; Jiang and Leyton-Brown 2013].

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Beyond computation of equilibria, another significant thread of research in algorithmic game theory has been the study of the *quality* of equilibria, often as measured by the social welfare of the equilibrium or its ratio to the social welfare of the socially optimal outcome (c.f. the extensive literature on the price of anarchy (PoA) [Nisan et al. 2007]). Given that we know it is possible to efficiently compute some CE and CCE, it is natural to ask *how good* an equilibrium we get as the output of such a procedure. For example, do existing efficient dynamics find the best such equilibria, or at least ones that approximately optimize the social welfare?

In their notable work, [Papadimitriou and Roughgarden 2008] show that determining a *socially optimal* CE is NP-hard in a number of *succinct* multiplayer games. This hardness result leaves open the question of computing *near-optimal* CE/CCE, i.e., whether there exist efficient algorithms that compute CE/CCE with welfare at least, say, α times the optimal, for a nontrivial approximation ratio $\alpha \leq 1$. This question forms the basis of our work [Barman and Ligett 2015].

Technical Aside (succinct games): In general multiplayer games the size of the normal form representation, N, is exponentially large in the number of players; one can compute a CE/CCE that optimizes a linear objective by solving a linear program of size polynomial in N, and hence the computational complexity of equilibrium computation is not interesting for general games. However, most games of interest—such as graphical games, polymatrix games, congestion games, and anonymous games—admit a succinct representation (wherein the above-mentioned linear program can be exponentially large in the size of the representation), and hence it is such succinctly representable games that we (and previous works) study.

Results: In [Barman and Ligett 2015] we establish that, unless P = NP, there does not exist any efficient algorithm that computes a CCE in succinct multiplayer games with welfare better than the *worst possible* CCE. We also extend the hardness result to approximate CCE. Therefore, while one can efficiently compute an approximate CCE in succinct multiplayer games, one cannot provide any nontrivial welfare guarantees for the resulting equilibrium (unless P = NP). Furthermore, these hardness results also hold specifically for potential games (generally considered to be a very tractable class of games), and persist even in settings where the gap between the best and worst equilibrium is large. It is relevant to note that these complexity barriers provide new motivation for studying the price of anarchy (the quality of the worst equilibrium) for CE and CCE, since generally that is the best thing we can hope to compute.

Our work also complements these hardness results by developing an algorithmic framework for computing an approximate CCE with welfare that is additively close to the optimal. This framework establishes a sufficient condition under which the above-mentioned complexity barriers can be circumvented. In particular, we show that if in a given game we can efficiently obtain an *additive* approximation for a *modified-welfare maximization problem*, then we can efficiently compute an approximate CE with high welfare. The modified welfare under consideration can be thought of as a Lagrangian corresponding to the equilibrium constraints. We instantiate this algorithmic framework to compute a high-welfare approximate CCE in *aggregative games*.

In the interest of space, the theorems presented below only address CCE; analogous results hold for CE. Also, details of the above mentioned positive results appear in [Barman and Ligett 2015].

2. PROBLEM DEFINITIONS AND RESULTS

We consider games with n players and m actions per player. Write A_p to denote the set of actions available to the pth player and A to denote the set of action profiles, $A := \prod_p A_p$. The (normalized) utility of player p is denoted by $u_p : A \to [0,1]$. We use $w(a)$ to denote the welfare of action profile $a \in A$, $w(a) = \sum_p u_p(a)$. Given a distribution x over the set A, we use $u_p(x)$ and $w(x)$ to denote the expected utility of player p and the expected welfare, respectively.

Definition 2.1 *ε-Coarse Correlated Equilibrium.* A probability distribution x over the action profiles A is said to be an ε-coarse correlated equilibrium if for every player p and every action $i \in A_p$ we have $\sum_{a \in A}[u_p(i, a_{-p}) - u_p(a)]x(a) \leq \varepsilon$.

The definition of a CCE is obtained by setting $\varepsilon = 0$ in the above inequality. Next we establish the hardness of the decision problem NT which is formally defined below. Note that acronym NT stands for nontrivial.

Definition 2.2 NT. Let Γ be an n-player m-action game with a succinct representation. NT is defined to be the problem of determining whether Γ admits a CCE x such that $w(x) > w(x')$. Here x' denotes the worst CCE of Γ, in terms of social welfare w.

To prove the subsequent theorem we start with a succinct n-player m-action game G from a class of games in which computing a welfare-maximizing action profile is NP-hard.[1] We reduce the problem of determining an optimal (welfare maximizing) action profile, say a^*, in G to solving NT in a modified succinct game G', which is obtained by providing each player in G with an additional action. Intuitively, the reduction works by ensuring that a^* is an optimal CCE in G' and any CCE with welfare better than the worst possible one can in fact be used to determine a^*; see [Barman and Ligett 2015] for a complete proof.

THEOREM 2.3. NT *is* NP*-hard in succinct multiplayer games.*

The hardness of NT implies that, under standard complexity-theoretic assumptions, it is impossible to efficiently compute a CCE that achieves a nontrivial approximation guarantee in terms of social welfare.

We also establish the hardness of computing an *approximate* CCE that has high social welfare. Specifically, we consider the problem of computing a $\frac{1}{2n^3}$-CCE with welfare $(1 + \frac{1}{n})$ times better than the welfare of the worst CCE. Note that there exist regret-based dynamics (cf. [Young 2004]) that converge to the set of ε-CCE in time polynomial in $1/\varepsilon$. Therefore, in polynomial time we can compute *a* $\frac{1}{2n^3}$-CCE. But, as the following theorem shows, it is unlikely that we can efficiently find a $\frac{1}{2n^3}$-CCE with any nontrivial welfare guarantee. Below we use acronym ANT to refer to the problem of determining an approximate CCE with nontrivial welfare.

[1] Multiple examples of such classes of games are given in [Papadimitriou and Roughgarden 2008].

Definition 2.4 ANT. Let Γ be an n-player m-action succinct game. ANT is defined to be the problem of determining whether there exists a $\frac{1}{2n^3}$-CCE x in Γ such that $w(x) \geq (1 + \frac{1}{n})w(x')$, where x' denotes the worst CCE of Γ, in terms of social welfare w.

Theorem 2.5. *In succinct multiplayer games,* ANT *is* NP*-hard under randomized reductions.*

Hardness results analogous to Theorem 2.3 and 2.5 hold for CE as well. Note that a classical interpretation of a CE is in terms of a mediator who has access to the players' payoff functions and who draws outcomes from a correlated equilibrium's joint distribution over player actions and privately recommends the corresponding actions to each player. The equilibrium conditions ensure that no player can benefit in expectation by unilaterally deviating from the recommended actions. Therefore, the problem we study here is exactly the problem that a mediator faces if she wishes to maximize social welfare.

In addition, it is shown in [Barman and Ligett 2015] that the above mentioned hardness results also hold specifically for potential games. This follows from the fact that the reduction used in the proofs of Theorem 2.3 and Theorem 2.5 gives us a potential game.

3. OPEN PROBLEMS

Overall, this work establishes a notable dichotomy: while one can efficiently compute an approximate CE and CCE, one cannot provide any nontrivial welfare guarantees for the resulting equilibria, unless P = NP. A number of interesting questions follow.

A relevant extension is to show that such hardness results hold even in specific classes of succinct games such as polymatrix games and graphical games. In terms of positive results, it would be interesting to determine additional classes of games which (like aggregative games) admit efficient computation of high-welfare CE and CCE. Developing dynamics that quickly converge to high-welfare CE/CCE in particular classes of games also remains an interesting direction for future work.

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