# Combinatorial Cost Sharing

SHAHAR DOBZINSKI
Weizmann Institute of Science
and
SHAHAR OVADIA
Weizmann Institute of Science

---

We introduce a combinatorial variant of the cost sharing problem: several services can be provided to each player and each player values every combination of services differently. A publicly known cost function specifies the cost of providing every possible combination of services. A combinatorial cost sharing mechanism is a protocol that decides which services each player gets and at what price. We look for dominant strategy mechanisms that are (economically) efficient and cover the cost, ideally without overcharging (i.e., budget balanced). Note that unlike the standard cost sharing setting, combinatorial cost sharing is a multi-parameter domain. This makes designing dominant strategy mechanisms with good guarantees a challenging task.

We present the Potential Mechanism – a combination of the VCG mechanism and a well-known tool from the theory of cooperative games: Hart and Mas-Colell's potential function. The potential mechanism is a dominant strategy mechanism that always covers the incurred cost. When the cost function is subadditive the same mechanism is also approximately efficient. Our main technical contribution shows that when the cost function is submodular the potential mechanism is approximately budget balanced in three settings: supermodular valuations, symmetric cost function and general symmetric valuations, and two players with general valuations.

---

## 1. INTRODUCTION

In their classic paper Littlechild and Owen [Littlechild and Owen 1973] consider the problem of fairly dividing runways maintenance costs among airlines. In this problem, each aircraft has a certain size and the cost of maintaining the runway depends on the size of the largest aircraft that uses the runway. Littlechild and Owen propose the following cost allocation scheme: first divide the cost of serving only the smallest airplane among all airlines. Then, divide the incremental cost for the second smallest airplane equally among all the airlines but the airline with the smallest aircraft. Continue thus until finally the incremental cost of the largest aircraft is divided among the airlines that use such aircraft.

Littlechild and Owen's work is an arch-typical example of a cost sharing problem, where the cost of a resource has to be divided among the participants. Cost sharing was extensively studied in game theory, and in fact, Littlechild and Owen show that their method coincides with the *Shapley value* [Shapley 1953].

In our work, we are interested in dominant-strategy mechanism for sharing the cost. The influential paper of Moulin and Shenker [Moulin and Shenker 2001] have shown how to use the Shapley values in order to achieve a truthful cost-sharing mechanism: serve the largest set of players such that every player in this set is willing to pay his Shapley value. They prove that for every submodular cost function it holds that the *Shapley value mechanism* is groupstrategyproof – no set of players can misreport in order to pay less. Furthermore, the cost of the service

is always covered.

The problem of cost sharing was extensively studied in both economics and algorithmic game theory and many variants were suggested and analyzed (e.g., [Moulin 1999; Moulin and Shenker 2001; Roughgarden and Sundararajan 2009; Bleischwitz et al. 2007; Bleischwitz and Schoppmann 2008; Deb and Razzolini 1999; Mehta et al. 2007; Hashimoto and Saitoh 2015]). Almost all the above variants considered the case of a single good, perhaps with different possible levels of services. One is naturally led to consider the following generalization of [Littlechild and Owen 1973]: what if there are several runways that can be potentially be constructed, or maybe we consider simultaneously building several distant airports? The preferences of an airline become much more complicated now, for example, a long runway in New York is less attractive if all London runways are much smaller.

In our paper we attempt to fill this lacuna and introduce combinatorial cost sharing, where multiple goods can be provisioned and both costs and preferences depend on the selected combination of goods. While combinatorial cost sharing is a natural generalization of the basic cost sharing scenario, from a technical perspective it is radically different as we leave the relatively safe single parameter world and cross the bridge to the realm of multi parameter mechanism design. Nevertheless, we will see that good mechanisms for combinatorial cost sharing do exist.

## 2. THE MODEL

The standard cost sharing setting (from now on, "simple cost sharing") involves a set $N$ of players ($|N| = n$), where the value of player $i$ is $v_i$ if player $i$ receives a usage permission and 0 otherwise. A known cost function $C : 2^N \to \mathbb{R}^+$ specifies the cost of serving each subset of the players. The goal is to decide which players to serve and for what prices.

In *combinatorial cost sharing* we have $n$ players as before, but now there are several public goods that can be constructed. We present two formulations of combinatorial cost sharing. The first is more direct formulation which might help the reader to digest the setting more easily. The second formulation – which is the one that is studied throughout the paper – is equivalent in power but is notationally more involved. We use it since it makes the technical proofs more readable.

*A First Attempt.* As in simple cost sharing, there is a set $N$ which consists of $n$ players, but now there is a set $M$ of public goods (for example, a pool, a gym, etc.). Players might have complicated preferences over the goods in $M$ (e.g., a combined membership for the pool and the gym might be more valuable than the sum of the values of each membership alone), thus the priveate valuation of player $i$ is $v_i : 2^M \to \mathbb{R}$. Note that this assumes that there are no externalities in the sense that value of each player is determined only by the goods he is served (in particular, the value does not depend on the other players who use these services).

Let $C' : (2^N)^m \to \mathbb{R}$ be a known function that specifies the cost of every possible combination of services. For example, $C(S_1, \ldots, S_n)$ is the cost of serving the first good in $M$ to the players in $S_1$ while serving the second good in $M$ to the players in $S_2$, and so on. We stress that we do not make any assumptions on $S_1, \ldots, S_n$ and in particular these sets are typically not disjoint.

*The Main Formulation.* The issue with the first formulation is that we often would like to assume that the cost function belongs to some standard class, e.g., $C'$ is submodular or subadditive. However, as defined $C'$ is not even a set function (its domain is $(2^N)^m$). We thus use a different formulation that is equivalent in power. Define – for notational convenience – for each player $i$ a set $M_i$ with $M_i \cap M_{i'} = \emptyset$ for $i \neq i'$, where each $j \in M_i$ represents a permission to consume a different good. For example, if $M$ is the set of public goods that can be constructed, we define for each player $i$ a set $M_i$, $|M_i| = |M|$, and think about the $j$'th item in $M_i$ as permission *for player $i$* to use the $j$'th public good in $M$. In particular player $i$ is never interested in items from $M_{i'}$, for $i \neq i'$. The private valuation of player $i$ is $v_i : 2^{M_i} \to \mathbb{R}$. The cost function $C : 2^{M_1} \times \cdots \times 2^{M_n} \to \mathbb{R}$ specifies the cost of every possible combination of services. Note that it is straightforward to express every cost function in the first formulation as a cost function in the main formulation. In particular, $C$ is now a set function (the set of items is $M_1 \cup \cdots \cup M_n$ – recall that $M_i \cap M_j$ for $i \neq j$), so standard notions such as subadditivity and submodularity are defined in the usual sense.

## 3.   COMBINATORIAL COST-SHARING MECHANISMS

A (direct) mechanism for combinatorial cost sharing problem receives as input the valuations of the players and outputs an allocation of services $\overrightarrow{ALG}$ where $ALG_i \subseteq M_i$ is the set of services provided to player $i$. Moreover for every player $i$ the mechanism specifies his payment $p_i$. It is standard to assume that the mechanism is individually rational (for every $i$, $p_i \leq v_i(ALG_i)$), $p_i \geq 0$ (no positive transfers) and moreover $p_i = 0$ if player $i$ is not served.

Work on cost sharing in the AGT community mostly focuses on incentive compatible mechanisms, either dominant strategy or groupstrategyproof, that at the very least always cover the cost. That is, in an instance where the mechanism outputs an allocation $\overrightarrow{ALG}$ we require to have $C(\overrightarrow{ALG}) \leq \Sigma_i p_i$. Ideally, we will also not overcharge the players, at least not by much: a mechanism is $\beta$-budget balanced if in every instance $C(\overrightarrow{ALG}) \leq \Sigma_i p_i \leq \beta \cdot C(\overrightarrow{ALG})$.

We look for mechanisms that are economically efficient. Following [Roughgarden and Sundararajan 2009], we look for mechanisms that minimize the *social cost*, $\pi(\overrightarrow{S}) = C(\overrightarrow{S}) + \Sigma_{i \in N}[v_i(M_i) - v_i(S_i)]$, which is the construction cost plus the "lost value" from not providing all the services. Minimizing the social cost has some appealing properties, for example, the social cost and the social welfare induce the same order on allocations. Moreover, additive approximations to the social welfare imply multiplicative guarantees on the social cost. We refer the interested reader to the paper for a complete discussion.

The simple cost sharing literature is rich in beautiful results, but the jewel in the crown is probably the Shapley value mechanism [Moulin and Shenker 2001], which is a groupstrategyproof mechanism that exactly shares the cost whenever $C$ is a submodular function [Moulin and Shenker 2001]. Roughgarden and Sundararajan [Roughgarden and Sundararajan 2009] show that it gives an approximation ratio of $\mathcal{H}_n = \Sigma_{i=1}^n \frac{1}{n}$ to the social cost. It is known that the approximation ratio of any mechanism that always covers the cost is $\Omega(\log n)$ and that this is true for every

dominant-strategy mechanism. [Dobzinski et al. 2008][1].

## 4.   COST RECOVERING MECHANISMS AND THE POTENTIAL MECHANISM

The simple cost sharing domain is a single parameter one, where the private information of every player consists of one number. Thus, to design a dominant strategy mechanism one can focus on the quite powerful family of monotone algorithms. In fact, the literature contains powerful techniques for designing groupstrategyproof mechanisms, for various notions of groupstrategyproofness (e.g., the Moulin family of mechanisms [Moulin 1999] and acyclic mechanisms [Mehta et al. 2007]).

In contrast, the combinatorial cost sharing domain is a multi-parameter one. The difficulty of designing useful mechanisms for multi-parameter domains is well known. The root of evil is the lack of general design techniques except the VCG family. For example, if the domain is unrestricted, then the only possible dominant strategy mechanisms are affine maximizers [Roberts 1979], a slight variation of VCG mechanisms. In general, more restricted domains as ours do allow for non VCG mechanisms, but the VCG family remains the main tool at our disposal.

However, while VCG is effective for welfare maximization, in cost sharing settings we also need to cover the construction cost. Unfortunately, conventional wisdom has it that the revenue of the VCG mechanism is uncontrollable and tends to be low[2] [Ausubel and Milgrom 2006]. The main technical contribution of this paper challenges this – we do manage to "tame" the VCG beast and obtain VCG based mechanisms that are approximately budget balanced.

For simplicity, we start by constructing VCG based mechanisms that always cover the cost for simple cost sharing, so the valuation of each player $i$ can be described by a single number: $v_i$ if served and 0 otherwise. In general, affine maximizers can have both (multiplicative) player weights and (additive) allocations weights. The former does not seem to be very useful, so we focus on affine maximizers of the form:

$$\arg\max_{S \subseteq N} \sum_{i \in S} v_i - H(S) \tag{1}$$

where $H : 2^N \to \mathbb{R}$ is a function that does not depend on the $v_i$'s. If $S$ is the allocation that maximizes (1) for the valuation profile $v$, then the payment of player $i$ is:

$$p_i = \sum_{j \in S_{-i}} v_j - H(S_{-i}) - \left[ \sum_{j \in S - \{i\}} v_j - H(S) \right] \tag{2}$$

where $S_{-i}$ is the allocation that maximizes (1) when the valuation of player $i$ is identically 0.

---

[1]This impossibility of [Dobzinski et al. 2008] is stated for budget balanced mechanisms, but the proof applies even to cost recovering mechanisms.

[2]Some papers attempt to control the revenue of VCG in simpler auction settings by rebating the players, e.g., [Moulin 2009; Guo and Conitzer 2009; Cavallo 2006]. Also relevant is the work of Blumrosen and Dobzinski [Blumrosen and Dobzinski 2014] which is the closest in spirit to ours (and in fact is the inspiration to our work). One of their results essentially provide a cost-recovering VCG based mechanism for the excludable public good problem. Their mechanism can be derived as a special case of our constructions.

When $H$ is the cost function $C$, we get a welfare maximizing mechanism. However, it is common for this mechanism to run a deficit, e.g., in the special case of excludable public good ($C(S) = 1$ for every $S \neq \emptyset$), if $v_i > 1$ for every player $i$, the revenue is 0.

Thus, to cover the incurred cost we need some other function $H \neq C$. Notice from the definition of $S_{-i}$ that $\sum_{j \in S_{-i}} v_j - H(S_{-i}) \geq \sum_{j \in S - \{i\}} v_j - H(S - \{i\})$. Hence, the payment of the $i$'th player $p_i$ is at least $H(S) - H(S - \{i\})$. A function $H$ with the property that for every set $S \subseteq N$ it holds that $\sum_{i \in S} H(S) - H(S - \{i\}) \geq C(S)$ will lead to a dominant-strategy mechanism for simple cost sharing that always collects payments that cover the incurred cost. For example, in the special case of excludable public good, we can choose $H(S) = \mathcal{H}_{|S|}$, so if a set $S$ is selected the marginal cost to $H$ of each player $i \in S$ is at least $H(S) - H(S - \{i\}) = \frac{1}{|S|}$ and the total payment is at least 1 (this special case was analyzed by Blumrosen and Dobzinski [Blumrosen and Dobzinski 2014]).

Interestingly, for *every* cost function $C$ the *potential function* of Hart and Mas-Colell [Hart and Mas-Colell 1989] satisfies this property. Hart and Mas-Colell suggested the potential function as an alternative simple way of defining the Shapley values. Fix some cost function $C$, and consider some function $P_C$ that assigns a value to every coalition $S \subseteq N$ and the cost function $C$. Suppose that $P_C$ is such that for every $S \subseteq N$ the marginal contributions of the players add up to the cost of the coalition: $\Sigma_{i \in S}(P_C(S) - P_C(S - \{i\})) = C(S)$. Hart and Mas-Collel show that $P_C$ exists and is unique. They term $P_C$ the potential function. This also gives an alternative definition for the Shapley values as it turns out that the Shapley value of player $i$ in a coalition $S$ is exactly its marginal contribution to $P_C$ ($Shapley_i(S) = P_C(S) - P_C(S - \{i\})$).

We generalize and adapt the potential function to our needs: the potential function as defined in [Hart and Mas-Colell 1989] considers cooperative games, i.e., the cost function defined on subsets of $N$. Our generalization considers allocations. Specifically, we define the marginal contribution of player $i$ to the allocation $(S_1, ..., S_n)$ by $P_C(S_1, ..., S_{i-1}, S_i, S_{i+1}, ..., S_n) - P_C(S_1, ..., S_{i-1}, \emptyset, S_{i+1}, ..., S_n)$.

We set the function $H$, which appears in (1) and (2), to be our generalization of the potential function, and name the new mechanism, a VCG mechanism using the potential function, the *Potential Mechanism*. Notice that this gives a dominant strategy cost-recovering mechanism for *every* cost function $C$. We are also able to prove efficiency guarantees if the cost function $C$ is subadditive. Combined together, we get the following general result:

**Theorem:** Let $C$ be a subadditive cost function. Then, the Potential mechanism always recovers the cost and provides an approximation ratio of $2\mathcal{H}_n$ to the social cost. If $C$ is submodular (or even XOS) the approximation ratio improves to $\mathcal{H}_n$.

Again, the approximation ratio is essentially tight due to the impossibility result of [Dobzinski et al. 2008].

## The Main Result

The main technical effort of this work is in identifying three settings in which the potential mechanism is not only efficient and cost recovering, but also budget

balanced.

**Theorem:** Let $C$ be a submodular cost function. The Potential Mechanism is $\mathcal{H}_n$-budget-balanced and provides an approximation ratio of $\mathcal{H}_n$ to the social cost in each of the following settings:

Supermodular valuations.

General symmetric valuations and player-wise symmetric cost function[3].

Two players ($n = 2$) with general valuations.

## 5. OPEN QUESTIONS

We have presented a novel way to share the cost among participants. Yet, we do leave many exciting questions open.

**Computational Issues.** We focused on proving existence of mechanisms with good guarantees for combinatorial cost sharing. Of course we would like to have a mechanism that both runs in polynomial time and has good guarantees, such as economic efficiency and budget balance. We do not know whether such a mechanism exists in general, however, we note that in the case of supermodular valuations and a submodular cost function the potential mechanism is computationally efficient whenever the value of the potential function is easy to compute.

**The Performance of the Potential Mechanism.** The potential mechanism guarantees to cover the cost and to provide approximation ratio of $2\mathcal{H}_n$ to the social cost for every subadditive cost function. It is not clear whether the mechanism is overcharging when considering a subadditive cost function (or even an $XOS$ cost function) that is not submodular. A first step to make in order to understand the guarantees of the potential mechanism is to determine the overcharging of the mechanism for a simple cost sharing when the cost function is subadditive.

**The Power of GSP vs. Strategyproof Mechanisms.** The potential mechanism, which is a VCG-based mechanism, is known to be vulnerable to strategic behaviors of groups of players. We know very little about groupstrategyproof mechanisms for combinatorial cost sharing. In the paper we provide a mechanism that guarantees a poor approximation ratio of $\Omega(n)$. Are there groupstrategyproof mechanisms with better guarantees?

**Impossibilities for Combinatorial Cost Sharing.** In light of the impossibility result of [Dobzinski et al. 2008], which provides a lower bound of $\Omega(\log n)$ to the approximation ratio of the social cost in simple cost sharing, we are looking for similar impossibilities results for combinatorial cost sharing, in particular for the setting of supermodular valuations and submodular cost function (recall that the potential mechanism is $\mathcal{H}_n$-budget-balanced and provides an approximation ratio of $\mathcal{H}_n$ to the social cost in this case). Is there a mechanism which is $\beta$-budget-balanced and provides approximation ratio of $\rho$ to the social cost where $\beta, \rho < \mathcal{H}_n$?

---

[3]A valuation is symmetric if $v_i(S) = v_i(T)$ whenever $|S| = |T|$. A cost function is player-symmetric if $C(\vec{S}) = C(\vec{T})$ whenever $|S_i| = |T_i|$ for all $i$.

REFERENCES

AUSUBEL, L. M. AND MILGROM, P. 2006. The lovely but lonely vickrey auction. In *Combinatorial Auctions, chapter 1*. MIT Press.

BLEISCHWITZ, Y., MONIEN, B., AND SCHOPPMANN, F. 2007. To be or not to be (served). In *International Workshop on Web and Internet Economics*. Springer, 515–528.

BLEISCHWITZ, Y. AND SCHOPPMANN, F. 2008. Group-strategyproof cost sharing for metric fault tolerant facility location. In *International Symposium on Algorithmic Game Theory*. Springer, 350–361.

BLUMROSEN, L. AND DOBZINSKI, S. 2014. Reallocation mechanisms. In *Proceedings of the Fifteenth ACM Conference on Economics and Computation*. EC '14. ACM, 617–617.

CAVALLO, R. 2006. Optimal decision-making with minimal waste: Strategyproof redistribution of vcg payments. In *Proceedings of the fifth international joint conference on Autonomous agents and multiagent systems*. ACM, 882–889.

DEB, R. AND RAZZOLINI, L. 1999. Voluntary cost sharing for an excludable public project. *Mathematical Social Sciences 37,* 2, 123–138.

DOBZINSKI, S., MEHTA, A., ROUGHGARDEN, T., AND SUNDARARAJAN, M. 2008. Is shapley cost sharing optimal? In *International Symposium on Algorithmic Game Theory*. Springer, 327–336.

GUO, M. AND CONITZER, V. 2009. Worst-case optimal redistribution of vcg payments in multi-unit auctions. *Games and Economic Behavior 67,* 1, 69–98.

HART, S. AND MAS-COLELL, A. 1989. Potential, value, and consistency. *Econometrica: Journal of the Econometric Society*, 589–614.

HASHIMOTO, K. AND SAITOH, H. 2015. Strategy-proof cost sharing under increasing returns: Improvement of the supremal welfare loss. *Games and Economic Behavior 89*, 101–121.

LITTLECHILD, S. C. AND OWEN, G. 1973. A simple expression for the shapley value in a special case. *Management Science 20,* 3, 370–372.

MEHTA, A., ROUGHGARDEN, T., AND SUNDARARAJAN, M. 2007. Beyond moulin mechanisms. In *Proceedings of the 8th ACM Conference on Electronic Commerce*. ACM, 1–10.

MOULIN, H. 1999. Incremental cost sharing: Characterization by coalition strategy-proofness. *Social Choice and Welfare 16,* 2, 279–320.

MOULIN, H. 2009. Almost budget-balanced VCG mechanisms to assign multiple objects. *Journal of Economic Theory 144,* 1, 96 – 119.

MOULIN, H. AND SHENKER, S. 2001. Strategyproof sharing of submodular costs: budget balance versus efficiency. *Economic Theory 18,* 3, 511–533.

ROBERTS, K. 1979. Aggregation and revelation of preferences. *The Characterization of Implementable Choice Rules*, 321–348.

ROUGHGARDEN, T. AND SUNDARARAJAN, M. 2009. Quantifying inefficiency in cost-sharing mechanisms. *Journal of the ACM (JACM) 56,* 4, 23.

SHAPLEY, L. S. 1953. A value for n-person games. *Contributions to the Theory of Games 2,* 28, 307–317.